Some Aspects of General Relativistic Vortex Dynamics and Ertel's Identity

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According to Ertel's identity and Ertel's vorticity theorem some aspects on general relativistic vortex dynamics and relationships are discussed in detail.

Key words: Ertel's Potential Vorticity; Ertel's Identity; Relativistic Vortex Dynamics.

The general theory of relativity gives on absolutely covariant formulation of Helmholtz' laws of vorticity which is valid in arbitrary reference-systems. For small relative velocities u_i (with $u_i u^i \ll c^2$) these generally covariant laws deliver Helmholtz's first law for a vorticity ω_i in a rigidly rotating reference-system $\frac{\mathrm{d}}{\mathrm{d}t}(\omega_i - \Omega_i) = (\omega^l - \Omega^l) u_{l,i}$ with the angular velocity Ω^l of the rotation.

Sommerfeld [1] remarks in his "Mechanik der deformierbaren Medien": "Vom Standpunkt des Erdbeobachters muß man die Corioliskraft berücksichtigen, oder man muß von dem scheinbaren Wirbel zum absoluten, durch die Corioliskraft modifizierten Wirbel übergehen". In their great article "The Classical Field Theories", Truesdell and Toupin [2] study continuum mechanics in a Galilean space-time. The non-inertial systems of [2] become a very sophisticated problem.

General relativity is based on the Lorentz-Einsteinian space-time world with Einstein's reference-"mollusces". Then, the questions of non-inertial reference-systems have straightforward solutions by the "absolutely" covariant formulation of continuum dynamics. In its three-dimensional form the classical Helmholtzian laws are valid for small relative velocities (small in comparison to the velocity of light c) and in inertial systems. In rotating reference systems Helmholtz's first law will be modified by Coriolis-accelerations.

General Relativistic Euler Equations

The general theory of relativity is not only the theory of gravitation but also the theory of reference-systems in four-dimensional Riemannian space-time. The general principle of relativity means the general covariance of the physical equations in respect to all coordinate-transformations $\bar{X}^\nu = \bar{X}^\nu (X^\lambda)$ (with $\nu, \lambda = 1, 2, 3, 4$) in the four-dimensional world with Lorentzian signature. The transformations of the general form

$$\bar{X}^\nu = \bar{X}^\nu (X^i, X^4) \text{ (with } i = 1, 2, 3 \text{ and } X^4 = ct)$$

and reciprocally

$$X^\nu = X^\nu (\bar{X}^i, \bar{X}^4) \text{ (with } \bar{X}^4 = c\bar{t}\,)$$

involve the transformations of the reference-systems. In

$$\bar{g}\,\mu\,\nu = \frac{\partial X^i}{\partial \bar{X}^\mu}\frac{\partial X^k}{\partial \bar{X}^\nu}\, g_{ik} + 2\,\frac{\partial X^i}{\partial \bar{X}^\mu}\frac{\partial X^4}{\partial \bar{X}^\nu}\, g_{i4} + \frac{\partial X^4}{\partial \bar{X}^\mu}\frac{\partial X^4}{\partial \bar{X}^\nu}\, g_{44}$$

the second and the third term and $\bar{t}$-dependence of the $X^i = X^i(\bar{X}^k, \bar{t}\,)$ mean a change of reference-systems.

According to Einstein's principle of equivalence, a space-time with a Riemannian curvature-tensor $R^x_{\alpha\beta\lambda} \neq 0$ is a world with gravitational fields. The flat space-time $R^x_{\alpha\beta\lambda} = 0$ is a world without gravitation. In this latter case, the Minkowski metrics $g_{\mu\nu} = \eta_{\mu\nu}$ describe inertial systems with four-dimensional cartesian coordinates. A metric

$$g_{ik} = \delta_{ik}\,, \quad \bar{g}_{i4} = \frac{\partial X^\mu}{\partial \bar{X}^i}\frac{\partial X^\nu}{\partial \bar{X}^4}\, \mu_{\mu\nu}\,,$$

$$\bar{g}_{44} = \frac{\partial X^\mu}{\partial \bar{X}^4}\frac{\partial X^\nu}{\partial \bar{X}^4}\, \eta_{\mu\nu}$$

describes with three-dimensional cartesian coordinates non-inertial reference system m^{in} a flat space-time and implies accelerations of the Coriolis- and the centrifugal-types: We write our equations with help of the mathemat-

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ical symbolism given in Einstein's book [3], but we choose the signature $\{+1,+1,+1,-1\}$. Then, $u^\nu=\frac{dx^\nu}{d\tau}$ (with the proper-time differential $d\tau=\frac{1}{c}\sqrt{-g_{\mu\nu}\,dx^\mu\,dx^\nu}$) is the four-vector of velocity with the norm $u^\nu=u_\nu=-c^2$.

The total derivation $\frac{d}{d\tau}A$ means

$$\frac{d}{d\tau}A=A,_\nu\,u^\nu=A,_\nu\,\frac{dx^\nu}{d\tau}$$

$$\left(\text{with } A,_\nu=\partial_\nu\,A=\frac{\partial}{\partial x^\nu}A\right).$$

The Covariant derivation of a vector is

$$A^\nu;_\lambda=A^\nu,_\lambda+\Gamma^\nu_{\mu\lambda}\,A^\mu\,,$$

and of a co-vector

$$A_\nu;_\lambda=A_\nu,_\lambda-\Gamma^\mu_{\nu\lambda}\,A_\mu$$

with the Christoffel-affinities

$$\Gamma^\mu_{\nu\lambda}=\frac{1}{2}\,g^{\mu\alpha}\,(-g_{\nu\lambda,\alpha}+g_{\lambda\alpha,\nu}+g_{\alpha\nu,\lambda})\,.$$

The Riemannian tensor is

$$R^\alpha_{\mu\nu\lambda}=-\Gamma^\alpha_{\mu\nu,\lambda}+\Gamma^\alpha_{\nu\lambda,\nu}-\Gamma^\alpha_{\tau\lambda}\,\Gamma^\tau_{\mu\nu}+\Gamma^\alpha_{\tau\nu}\,\Gamma^\tau_{\mu\lambda}\,.$$

The relativistic energy-momentum-tensor (matter-tensor) of a perfect fluid is [3]

$$T_{\mu\nu}=\rho_\mu\,u_\nu-\frac{p}{c^2}\,u_\mu\,u_\nu+p\,g_{\mu\nu}=\sigma u_\mu\,u_\nu+p\,g_{\mu\nu}$$

(ρ = matter-density, p = pressure), and the general-relativistic Euler equations result from the dynamical equations for

$$T^\nu_\mu;_\nu=(\sigma\,u^\nu);_\nu\,u_\mu+\sigma\,u_{\mu;\nu}\,u^\nu+p,_\mu=0\,,\qquad(1)$$

or with $u^\mu\,u_\mu=-c^2$

$$c^2\,(\sigma\,u^\nu);_\nu=\frac{d}{d\tau}\,p\,,\qquad(1\,a)$$

$$\sigma\,u_{\mu;\nu}+\frac{u_\mu}{c^2}\,\frac{dp}{d\tau}+p,_\mu=\sigma\,\frac{D}{D\tau}\,u_\mu$$
$$+\frac{u_\mu}{c^2}\,\frac{dp}{d\tau}+p,_\mu=0\,.\qquad(1\,b)$$

For $u_i\,u^i\ll c^2$, that is for small relativistic velocities (in comparison to the velocity of light c), we have the three-dimensional Euler equation in a non-inertial system: (1 a) gives

$$\frac{dp}{dt}+(\rho\,u^i);_i=0\qquad(2\,a)$$

and (1 b) gives with $u^4\approx c$

$$\frac{d}{dt}\,u_i=-\frac{1}{2}\,g_{44,i}\,c^2$$
$$+(g_{4i,l}-g_{4l,i})\,c u^l-\frac{1}{\rho}\,p,_i\,.\qquad(2\,b)$$

In (2 b) the form

$$(g_{4i,l}-g_{4l,i})\,c\,u^l\qquad(3\,a)$$

means the Coriolis-acceleration and

$$-\frac{1}{2}\,g_{44,i}\,c^2\qquad(3\,b)$$

is the sum of Newtonian and centrifugal accelerations. We remark that (3 a) has the form of a three-dimensional Lorentz-force [4].

Ertel's Identity

The vorticity of four-velocities u_ν are defined by [5]

$$y_{\mu\nu}=u_{\mu;\nu}-u_{\nu;}+\left(\frac{Du}{D\tau}\frac{u_\nu}{c^2}-\frac{Du_\nu}{D\tau}\frac{u_\mu}{c^2}\right)$$
$$=\omega_{\mu\nu}+\frac{1}{c^2}\,(P_\mu\,u_\nu-P_\nu\,u_\mu)\qquad(4)$$

with the curl of the vector u_ν

$$\omega_{\mu\nu}=u_{\mu;\nu}-u_{\nu;\mu}=u_{\mu,\nu}-u_{\nu,\mu}\qquad(4\,a)$$

and the acceleration four-vector

$$P_\mu=\frac{Du_\mu}{D\tau}=\gamma_{\mu\nu}\,\frac{D^2x^\nu}{D\tau^2}\,.\qquad(4\,b)$$

The total time-derivations of $\omega_{\mu\nu}$ are

$$\frac{D}{D\tau}\,\omega_{\alpha\beta}=\omega_{\alpha\beta;\lambda}\,u\lambda=u_{\alpha;\beta\lambda}\,u^\lambda-u_{\beta;\alpha\lambda}\,u^\lambda$$
$$=(u_{\alpha;\beta\lambda}-u_{\beta;\lambda\alpha})\,u^\lambda$$
$$-(R_{x\alpha\beta\lambda}-R_{x\beta\alpha\lambda})\,u^x\,u^\lambda\,.\qquad(5)$$

The second term on the right side in (5) vanishes because of the symmetries of the Riemannian tensor $R_{x\alpha\beta\lambda}$. Further we have the identities

$$u_{\alpha;\lambda\beta}\,u^\lambda=u_{\alpha;\gamma}\,u^\lambda;_\beta+P_{\alpha;\beta}$$

and

$$u_{\alpha;\lambda}\,u^\lambda;_\beta=u_{\alpha;\lambda}\,\omega^\lambda_\beta+u_{\alpha;\lambda}\,u^\lambda_{\beta;}$$
$$=u^\lambda;_\beta\,\omega_{\alpha\lambda}+u^\lambda;_\beta\,u_{\lambda;\alpha}\,,$$

and therefore Ertel's identity [6–8] in relativistic form [9] (see also [11–15]) reads

$$\omega^\lambda_\alpha\,u_\beta;_\lambda-\omega^\lambda_\beta\,u_{\alpha;\lambda}=\omega^\lambda_\alpha\,u_{\lambda;\beta}-\omega^\lambda_\beta\,u_{\lambda;\alpha}\,.$$

For the curl $\omega_{\alpha\beta}$ the relativistic forms of Helmholtz's and Boltzmann's first law are valid:

$$\frac{\mathrm{D}}{\mathrm{D}\tau}\,\omega_{\alpha\beta} = \omega^{\lambda}_{\alpha}\, u_{\lambda;\beta} - \omega^{\lambda}_{\beta}\, u_{\lambda;\alpha} + P_{\alpha,\beta} - P_{\beta,\alpha}$$
$$= \omega^{\lambda}_{\alpha}\, u_{\beta;\lambda} - \omega^{\lambda}_{\beta}\, u_{\alpha;\lambda} + P_{\alpha,\beta} - P_{\beta,\alpha}\,.$$

The time-variation of the vorticity $\gamma_{\alpha\beta}$ becomes [9]

$$\frac{\mathrm{D}}{\mathrm{D}\tau}\,\gamma_{\alpha\beta} = \omega^{\lambda}_{\alpha}\, u_{\lambda;\beta} - \omega^{\lambda}_{\beta}\, u_{\lambda;\alpha} + P_{\alpha,\beta} - P_{\beta,\alpha}$$
$$+\frac{1}{\tau^2}\left(\frac{\mathrm{D}}{\mathrm{D}\tau}\,P_{\alpha}\, u_{\beta} - \frac{\mathrm{D}}{\mathrm{D}\tau}\,P_{\beta}\, u_{\alpha}\right).$$

These equations are the general relativistic form of Helmholtz' first vorticity-law. Helmholtz's second law means

$$\omega_{\alpha\beta,\,\lambda} + \omega_{\beta\lambda,\,\alpha} + \omega_{\lambda\alpha,\,\beta} = 0\,,$$

see [9].

Coriolis Forces

The three-dimensional vorticities are given by the axial vector

$$\omega^{i} = \frac{1}{2}\,\varepsilon^{ikl}\,\omega_{kl} \quad (\text{with } \omega_{kl} = u_{k,\,l} - u_{l,\,k}\,). \qquad (6)$$

Helmholtz' first law in three-dimensional formulation for small relative velocities u_i (in comparison with the velocity of light c, that is $u^i\, u_i \ll c^2$) but in non-inertial reference-systems reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\omega_{ik} = \omega^{l}_{i}\, u_{l,\,k} - \omega^{l}_{k}\, u_{l,\,i}$$
$$\neq [(g_{4i,\,l} - g_{4l,\,i})\, u^{l} c]_{,k}$$
$$- [(g_{4k,\,l} - g_{4l,\,k})\, u^{l} c]_{,i}\,.$$

In rigidly rotating reference-systems with a constant angular velocity

$$\Omega^{i} = \varepsilon^{ikl}\,\Omega_{kl} \qquad (\text{with } \Omega_{kl} = -\Omega_{lk})$$

we have

$$g_{4i} = \frac{1}{c}\,\Omega_{ik}\, X^{k}\,. \qquad (7)$$

With (7) and (6) the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\omega_{ik} = (\omega^{l}_{i} - 2\,\Omega^{l}_{i})\, u_{l,\,k} - (\omega^{l}_{k} - 2\,\Omega^{l}_{k})\, u_{l,\,i}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\omega_{i} = (\omega^{l} - \Omega^{l})\, u_{l,\,i} \qquad (8)$$

result from (6). Equation (8) is the first Helmholtz-law in rigidly rotating systems like the earth. The meaning of (8) is clear. Instead of the Galilean-vorticity ω_i we use the difference $\omega_i - \Omega_i$ in rigidly rotating systems:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\omega_i - \Omega_i) = (\omega^{l} - \Omega^{l})\, u_{l,\,i}\,. \qquad (9)$$

We remark that a "Machian dynamics" with gravitationally induced Coriolis-forces gives (9) for the vorticity in a rotating bucket, also. Then the mass of such a Newtonian bucket is very small in respect to the mass of the universe [10].

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